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Notes

A Note on Rates of Convergence in Estimating Absolutely Continuous Distribution Functions*

INTRODUCTION

Let $\{P_n\}_0^\infty$ be a sequence of polynomials that obeys the following triple recurrence formula:

$$(*) \qquad \begin{aligned} &P_0(x) = 1, \qquad P_1(x) = x - a_0, \\ &P_{n+1}(x) = (x - a_n)\, P_n(x) - b_n P_{n-1}(x), \qquad n \geqslant 1. \end{aligned}$$

Here, $\{a_n\}_0^\infty$ is a real sequence and $\{b_n\}_1^\infty$ is a positive sequence. It is well known that under these assumptions (see [1]) there is a distribution function ψ such that the polynomials $\{P_n\}_0^\infty$ are orthogonal with respect to ψ, i.e., $\int_{-\infty}^{+\infty} P_i(x)\, P_j(x)\, d\psi(x) = k_i \cdot \delta_{ij}$, $k_i \neq 0$, $i, j = 0, 1, 2,\ldots$. (*We define a distribution function to be a nondecreasing, left continuous, real function* ψ *defined on* $(-\infty, +\infty)$ *and such that* $\psi(-\infty) = 0$, $\psi(+\infty) = 1$, *and* $\int_{-\infty}^{+\infty} x^k\, d\psi(x)$ *exists and is finite for* $k = 0, 1, 2,\ldots$.)

The sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ determine ψ, however, in general, they do not determine ψ uniquely. The nonunique cases correspond to undetermined moment problems (see [5] for a discussion of these problems and for conditions that determine ψ uniquely). Recently, there have been a number of results concerning the relationship between the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ and the distribution function ψ. In particular, the papers [2, 3] are concerned with the construction of ψ from the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$. These results describe ψ in terms of the properties of the complete sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$. However, in practice, it is likely that one does not construct ψ, but instead, one approximates ψ by constructing a distribution function φ using only the first N terms of these sequences for some finite N. This leads us to consider the following question: *Given finite real sequences* $\{a_n\}_0^N$ *and* $\{b_n\}_1^N$, $b_n > 0$, $n = 1, 2,\ldots, N$, *and two distribution functions* ψ *and* φ *such that the polynomials* $\{P_n\}_0^N$ *given by* (*) *are orthogonal with respect to both* ψ *and* φ, *then, how are* ψ *and* φ *related?*

Since the sequences $\{a_n\}_0^N$ and $\{b_n\}_1^N$ uniquely determine the first $2N + 1$ moments of the distribution ψ and φ, an equivalent questions is: *Given*

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two distribution functions ψ and φ with the same first $2N + 1$ moments, how are ψ and φ related?

In a certain sense, these questions were answered by both Stieltjes and Markoff when they independently proved the well-known Tchebycheff inequalities. These inequalities give pointwise bounds for the difference between any two such distribution functions ψ and φ. These pointwise bounds are sharp in the sense that for each real number a there are distribution functions φ_a and ψ_a such that $|\psi_a(a) - \varphi_a(a)|$ is as large as possible for all pairs of distribution functions taken from the set of distribution functions with the same first $2N + 1$ moments. However, these estimates are not sharp global estimates for the pointwise distance between ψ and φ, and hence, they do not yield sharp bounds for other measures of the distance between ψ and φ such as $\int_{-\infty}^{+\infty} | d\psi - d\varphi |$ and $\int_{-\infty}^{+\infty} | \psi - \varphi | \, dx$. It is these quantities that we estimate below. Our estimates are not given for general distribution functions, but instead are restricted to absolutely continuous distribution functions with continuous density functions having compact support.

In Section 2 we quote a preliminary result that is needed to obtain our estimates. The estimates of $\int | \psi - \varphi | \, dx$ and $\int | \psi' - \varphi' | \, dx$ are given in Section 3.

2. Preliminaries

The estimates that are obtained in Section 3 are only one form of a variety of such estimates that can be obtained by the same method. The exact form of the estimate depends on the type of approximation theorem that is used to obtain the estimate. In each case, the approximation is that of a continuous function by a polynomial of a fixed degree over a bounded interval. The different estimates result from different hypothesis about the smoothness of the continuous function. The assumptions that we make are those of the following theorem by Jackson [4, p. 56, 57]. It will be clear in Section 3 how other similar theorems will directly lead to other estimates.

THEOREM 1. (Jackson). *Let f be continuous on $[-1, +1]$ and suppose f' exists and is in class $\mathrm{Lip}_M\, \alpha$ on $[-1, +1]$. Then, for each integer $n \geqslant 2$, there are polynomials R_n and Q_n of degree $\leqslant n$ such that*

$$\sup_{x \in [-1, +1]} |f'(x) - Q_n(x)| \leqslant cM/n^{\alpha},$$

and

$$\sup_{x \in [-1, +1]} |f(x) - R_n(x)| \leqslant c^2 M/n(n-1)^{\alpha},$$

where $c = 1 + (\pi^2/2)$.

3. Estimates of $\int |\psi - \varphi| \, dx$ and $\int |\psi' - \varphi'| \, dx$

If a distribution function is constant outside of an interval $[A, B]$, then we say that the distribution function is *on* $[A, B]$. In such a case, we assume that the function is normalized to have value 0 for $x \leqslant A$, and value 1 for $x \geqslant B$.

The estimates given below are for functions on $[-1, +1]$. Similar results can be given for functions on any bounded interval $[A, B]$ by using a simple change of variable.

THEOREM 2. *Let ψ and φ be distribution functions on $[-1, +1]$, and suppose ψ' and φ' exist and are in the class $\mathrm{Lip}_M\, \alpha$ on $[-1, +1]$. Then, if ψ and φ have the same first $2N + 1$ moments (or, equivalently, the same orthogonal polynomials to degree N), then, for $N \geqslant 2$*

$$\int_{-1}^{+1} |\psi(x) - \varphi(x)| \, dx \leqslant \frac{4c^2 M}{(2N-1)(2N-2)^\alpha} .$$

Proof. The function $\psi - \varphi$ is continuous on $[-1, +1]$ and its derivative is of class $\mathrm{Lip}_{2M}\, \alpha$. Thus, by Theorem 1, there is a polynomial Q_{2N-1} of degree $\leqslant 2N - 1$ such that

$$\sup_{x \in [-1\ +1]} |[\psi(x) - \varphi(x)] - Q_{2N-1}(x)| \leqslant \frac{2Mc^2}{(2N-1)(2N-2)^\alpha} . \tag{1}$$

Next, we note that

$$\int_{-1}^{+1} [(\psi - \varphi) - Q_{2N-1}]^2 \, dx$$
$$= \int_{-1}^{+1} (\psi - \varphi)^2 \, dx - 2 \int_{-1}^{+1} (\psi - \varphi)\, Q_{2N-1} \, dx + \int_{-1}^{+1} Q_{2N-1}^2 \, dx, \tag{2}$$

and in this expression, we claim that $\int_{-1}^{+1} (\psi - \varphi)\, Q_{2N-1} \, dx$ has value zero. Our argument is as follows. Since ψ and φ have the same first $2N + 1$ moments, it follows that $\int_{-1}^{+1} P(x)\, d\psi(x) = \int_{-1}^{+1} P(x)\, d\varphi(x)$ for every polynomial $P(x)$ of degree $\leqslant 2N$. Thus, $\int_{-1}^{+1} P(x)[d\psi - d\varphi] = \int_{-1}^{+1} P(x)(\psi' - \varphi')\, dx = 0$ for all polynomials $P(x)$ of degree $\leqslant 2N$. Next, let Q_{2N} be any polynomial such that $Q_{2N}' = Q_{2N-1}$, and apply integration by parts to $\int_{-1}^{+1} (\psi - \varphi) Q_{2N-1} \, dx$. This gives

$$\int_{-1}^{+1} (\psi - \varphi)\, Q_{2N-1} \, dx = (\psi - \varphi)\, Q_{2N} \Big|_{x=-1}^{x=+1} - \int_{-1}^{+1} Q_{2N}(\psi' - \varphi') \, dx. \tag{3}$$

But $\psi(1) = \varphi(1) = 1$, $\psi(-1) = \varphi(-1) = 0$, and Q_{2N} has degree $\leqslant 2N$, and hence, each term on the right side of (3) has value zero. Therefore, our claim is established and (2) has the form

$$\int_{-1}^{+1} [(\psi - \varphi) - Q_{2N-1}]^2 \, dx = \int_{-1}^{+1} (\psi - \varphi)^2 \, dx + \int_{-1}^{+1} Q_{2N-1}^2 \, dx. \quad (4)$$

Combining Eqs. (1) and (4) we have

$$\begin{aligned} \int_{-1}^{+1} (\psi - \varphi)^2 \, dx &\leqslant \int_{-1}^{+1} \left[\frac{2c^2M}{(2N-1)(2N-2)^\alpha}\right]^2 dx \\ &= 2\left[\frac{2c^2M}{(2N-1)(2N-2)^\alpha}\right]^2. \end{aligned} \quad (5)$$

The result now follows since

$$\int_{-1}^{+1} |\psi - \varphi| \, dx \leqslant \left[\int_{-1}^{+1} (\psi - \varphi)^2 \, dx\right]^{1/2} \left[\int_{-1}^{+1} dx\right]^{1/2},$$

and combining this with (5) gives

$$\int_{-1}^{+1} |\psi - \varphi| \, dx \leqslant \frac{4c^2M}{(2N-1)(2N-2)^\alpha}.$$

The same techniques as those used in Theorem 2 (without integration by parts) give the following result.

THEOREM 3. *Let ψ and φ be distribution functions on* $[-1, +1]$ *and suppose ψ' and φ' exist and are in class* $\mathrm{Lip}_M\,\alpha$ *on* $[-1, +1]$. *Then, if ψ and φ have the same first $2N+1$ moments (or, equivalently, the same orthogonal polynomials to degree N), then for $N \geqslant 1$*

$$\int_{-1}^{+1} |\psi' - \varphi'| \, dx \leqslant 4cM/(2N)^\alpha.$$

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